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THE ORBITAL STABILITY OF THE TRAJECTORIES OF DYNAMIC SYSTEMS*

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An orbital stability criterion, generating Poincaré's criterion /1/ and the results of Hartman and Olech /2/, is derived. The application of this criteron is illustrated in the case of a two-dimensional dynamic system with an angular coordinate. The problem of the global asymptotic stability of the Lorenz system is considered.

Consider the system

$$dx/dt = f(x), \quad x \in R^n \tag{1}$$

where $f(x)$ is a twice continuously differentiable vector-valued function.

We shall say that a component x_j of the vector x is an angular coordinate if $f(x_1, \ldots, x_j, \ldots, x_n) \equiv f(x_1, \ldots, x_j + 2\pi, \ldots, x_n)$.

Let $x(t)$ be some trajectory of system (1), contained at $t \geqslant 0$ in a region $G \subset R^n$ which is bounded with respect to the non-angular coordinates. Henceforth we shall also assume that $f(x) \neq 0$ in the closure $\bar{G}$ of G.

We now introduce a symmetric non-singular matrix $H(x) = \| h_1, \ldots, h_n \|$, where $h_i(x)$ are twice continuously differentiable vector-valued functions, and a twice continuously differentiable vector-valued function $q(x)$ satisfying the inequality $f(x)^* q(x) \neq 0$, $\forall x \in \bar{G}$.

Let H_0 be a symmetric $(n \times n)$ matrix, $\lambda(x)$ a differentiable function, and t_j and ρ_j real sequences satisfying the conditions $\rho_j \leqslant \varkappa_1 < 0$, $t_{j+1} > t_j$, $t_{j+1} - t_j \leqslant \varkappa_2$, where $\varkappa_1$ and $\varkappa_2$ are numbers.

We will also put

$$\left(\frac{\partial H}{\partial x}, f\right) = \left\| \frac{\partial h_1}{\partial x} f, \ldots, \frac{\partial h_n}{\partial x} f \right\|, \quad f = f(x)$$

where $\partial h/\partial x$ is the Jacobian of the vector-valued function $h(x)$ at x.

Theorem 1. Assume that

$$\begin{gathered} \frac{1}{2} z^* \left(\frac{\partial H}{\partial x}, f\right) z + z^* H \frac{\partial f}{\partial x} z - \frac{z^* H f}{f^* q} \left[f^* \frac{\partial q^*}{\partial x} + q^* \frac{\partial f}{\partial x} \right] z \leqslant \\ \lambda z^* H z, \quad \forall z \in \{ z \mid z^* q(x(t)) = 0 \} \\ H = H(x(t)), \quad f = f(x(t)), \quad q = q(x(t)), \quad \Lambda = \Lambda(x(t)) \end{gathered} \tag{2}$$

Then, if the quadratic form $z^* H(x(t)) z$ is positive definite on the set $\{z \mid z^* q(x(t)) = 0\}$ and moreover

$$\Lambda_j = \int_{t_j}^{t_{j+1}} \lambda(x(t))\, dt \leqslant \rho_j \tag{3}$$

then the trajectory $x(t)$ is orbitally asymptotically stable.

If the quadratic form $z^* H(x(t_j)) z$ is non-degenerate on the set $\{z \mid z^* q(x(t_j)) = 0\}$, can take negative values and moreover

$$\begin{gathered} \Lambda_j \geqslant -\rho_j \\ z^* H(x(t)) z \geqslant z^* H_0 z, \quad \forall z \in \{ z \mid z^* q(x(t)) = 0 \} \end{gathered} \tag{4}$$

then the trajectory $x(t)$ is orbitally unstable.

Proof. Consider the set

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$$\Omega(\delta) = \bigcup_{t\geqslant 0} \{y \mid (y-x)^* H(x)(y-x) = \delta, \quad (y-x)^* q(x) = 0\}, \quad x = x(t)$$

Here δ is some sufficiently small number.

Fixing a point $y_0 \in \Omega(\delta)$, we investigate the surface $\Omega(\delta)$ in a fairly small neighbourhood of y_0. Since $y_0 \in \Omega(\delta)$, a number $t > 0$ exists such that

$$z^* H(x) z = \delta, \; z^* q(x) = 0, \; z = y_0 - x, \; x = x(t)$$

Let τ be a number near t. Then

$$x(\tau) \approx x(t) + f(x(t))(\tau - t)$$

We will now define a mapping (throughout, unless otherwise stated, $f = f(x)$, $K = K(x)$, $q = q(x)$, $x = x(t)$)

$$v(y_0) = y_0 + \alpha[f + Kz]$$

which carries the point y_0 into the hyperplane

$$\Phi = \left\{v \mid w^* \left[q + (\tau - t)\frac{\partial q}{\partial x} f\right] = 0\right\}, \quad w = v - (x + f(\tau - t))$$

in such a way that

$$w_0^* H(x + (\tau - t) f) w_0 \approx \delta, \; w_0 = v(y_0) - (x + (\tau - t) f) \tag{5}$$

The number α will be chosen so that $v(y_0) \in \Phi$, while the matrix K is chosen so as to satisfy (5). Clearly,

$$\alpha \approx \frac{f^* q - z^* \dfrac{\partial q}{\partial x} f}{f^* q + q^* K z} (\tau - t)$$

We are assuming here that $z(\tau - t)^{-1}$ is large. Hence it follows that a sufficient condition for (5) to be valid is that

$$\frac{1}{2} z^* \left(\frac{\partial H}{\partial x}, f\right) z + z^* H\left[K - \frac{f q^*}{f^* q} K - \frac{f f^*}{f^* q} \frac{\partial q^*}{\partial x}\right] z = 0 \tag{6}$$
$$\forall z \in \{z \mid z^* q(x(t)) = 0\}$$

It follows from (5) that a vector $l(y_0)$ normal to $\Omega(\delta)$ at the point y_0 can be determined as follows:

$$l(y_0) = l_1 - \frac{l_1^* l_2}{q^* l_2} q, \; l_1 = l_1(y_0) = 2(I - Q) H z, \quad Q = q q^* / |q|^2,$$
$$l_2 = l_2(y_0) = f + K z$$

Note that

$$\tfrac{1}{2} l(y_0) = (I - L_2)(I - Q) H z = (I - L_2) H z, \; L_2 = q l_2^* / q^* l_2$$

Therefore,

$$\frac{1}{2} l(y_0)^* f(y_0) \approx \left[f + \frac{\partial f}{\partial x} z\right]^* (I - L_2) H z \approx z^* H \left(I - \frac{f q^*}{f^* q}\right)\left(\frac{\partial f}{\partial x} - K\right) z$$

Hence, using (6), we see that

$$\frac{1}{2} l(y_0)^* f(y_0) \approx z^* \left\{\frac{1}{2}\left(\frac{\partial H}{\partial x}, f\right) + H \frac{\partial f}{\partial x} - H f \frac{1}{f^* q}\left(f^* \frac{\partial q^*}{\partial x} + q^* \frac{\partial f}{\partial x}\right)\right\} z \tag{7}$$

We can now show that the trajectory $y(t)$ of system (1) passing at time t through y_0 will satisfy the following inclusion relation to within $(\tau - t)^2$:

$$y(\tau) \in \Omega(\delta + (\tau - t)\, l(y_0)^* f(y_0)) \tag{8}$$

To that end, we observe that for small $(\tau - t)$ $y(\tau) \approx y(t) + f(y(t))(\tau - t)$. Hence the vector $y(\tau)$ lies, to within $(\tau - t)^2$, in the hyperplane L parallel to the hyperplane tangent to $\Omega(\delta)$ and passing through the point

$$y_0 + l(y_0) l(y_0)^* f(y_0) \mid l(y_0) \mid^{-2} (\tau - t)$$

It is also clear that L passes through the point $y_0 + u$ lying on the hyperplane

$$\{x \mid q(x(t))^*(x - x(t)) = 0\}$$

where

$$u = l_1(y_0) l(y_0)^* f(y_0) \mid l_1(y_0) \mid^{-2} (\tau - t)$$

Hence, using the relation $2(y_0 - x(t))^* H(x(t)) u = (\tau - t) l(y_0)^* f(y_0)$ and the fact that the vectors normal to L and to $\Omega(\delta + (\tau - t) l(y_0)^* f(y_0))$ at the point $y_0 + u$ are identical to within $(\tau - t)$, we obtain (8).

The inclusion relation (8), Eq.(7) and condition (2) of the theorem imply that for all $\tau \geqslant t$ one has $y(\tau) \in \Omega(\varphi(\tau))$, where $\varphi(\tau)$ is some continuous function such that

$$\varphi(\tau) \leqslant \delta \exp \int_t^\tau \lambda(x(t))\, dt$$

Using this inequality and conditions (3) and (4) of the theorem, and applying the standard Lyapunov technique /1, 2/, we obtain the assertion of the theorem.

Note that in the stable case, putting $q(x) = H(x) f(x)$, $\lambda(x) \equiv \text{const}$, Theorem 1 implies an assertion similar to Theorem 14.2 in /2/.

Now let us assume that the matrix in Theorem 1 has the form $H(x) = \mid f(x) \mid^2 I$, $\lambda(x) = \lambda_1(x) + \lambda_2(x)$, where λ_1 and λ_2 are eigenvalues of the matrix $(\partial f/\partial x + \partial f^*/\partial x)/2$ which satisfy the conditions $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$. We then obtain the following assertion from Theorem 1 and well-known results /2/:

Theorem 2. If a number $\varepsilon > 0$ exists such that for some solution $x(t) \in G$

$$\int_{t_j}^{t_{j+1}} [\lambda_1(x(t)) + \lambda_2(x(t))]\, dt \leqslant -\varepsilon, \quad \forall j \tag{9}$$

then $x(t)$ is orbitally asymptotically stable.

Theorem 2 may be viewed as a generalization, to some extent, of Poincaré's criterion /1/ and the Hartman-Olech theorem /2/.

Let us assume now that the set $\bar{G}$ is positively invariant and that $\bar{G}$ contains a unique asymptotically stable equilibrium state of system (1). In that case, using Theorem 2 and arguments from /2/, we obtain the following

Theorem 3. If for any solution $x(t) \in G$ inequality (9) is satisfied; then G is the domain of attraction of the stable equilibrium state.

It is also clear from Theorem 2 that if there is no equilibrium state in a positively invariant set $\bar{G}$, but inequality (9) is still true, then trajectories of system (1) situated in $\bar{G}$ will approach one another as $t \to +\infty$.

We will now consider some examples illustrating the application of Theorems 1-3.

Example 1. Consider the equation

$$\theta'' + \alpha\theta' + \varphi(\theta) = 0 \tag{10}$$

where α is a positive number, and $\varphi(\theta)$ is a twice differentiable 2π-periodic function with two zeros θ_1 and θ_2 in the set $[0, 2\pi)$. Eq.(10) describes the motion of a pendulum in a viscous medium /3/, the dynamics of a synchronous motor in its simplest idealization /4/, the operation of certain phase synchronization systems /5/, and the dynamics of Josephson junctions /6/.

Let $\varphi'(\theta_j) \neq 0$ and

$$\int_0^{2\pi} \varphi(\theta)\, d\theta < 0$$

Then it is well-known /3/ that a number $\alpha_{cr} > 0$ exists such that for $\alpha < \alpha_{cr}$ one can find in the phase space of the system

$$\theta' = \eta, \quad \eta' = -\alpha\eta - \varphi(\theta) \tag{11}$$

a positively invariant set $\bar{G}_1$, bounded with respect to the coordinate η, which is filled with circular motions /3, 7/. Moreover, G_1 will also contain a limit cycle of the second kind. Since $\lambda_1(x) + \lambda_2(x) = -\alpha < 0$ for system (11), it follows from Theorem 2 that a limit cycle of the second kind will be orbitally stable and G_1 is its domain of attraction.

On the other hand, if $\alpha < \alpha_{cr}$ the phase space of system (11) will contain a bounded positively invariant set $\bar{G}_2$ which contains a unique asymptotically stable equilibrium state /3/. It follows at once from Theorem 3 that G_2 is the domain of attraction of this state.

The only trajectories of system (11) for which the conditions of Theorem 2 do not all hold are saddle-point equilibrium states and the separatrices that approach them as $t \to +\infty$. (The condition that fails to hold here is $x(t) \in G$, where G does not contain equilibrium states). In the final analysis, therefore, the above-mentioned trajectories will be the boundaries of the domains of attraction of the stable equilibrium states and limit cycles of the second kind.

This result is well-known /3/ and can be derived by other, different methods. It is worth noting here that the use of Theorem 2 and 3 involves a minimum of calculations.

Example 2. Let us investigate the global asymptotic stability of the Lorenz system /8, 9/

$$x^{\cdot} = -d(x-y),\ y^{\cdot} = rx - y - xz,\ z^{\cdot} = -bz + xy \qquad d > 0,\ r > 1,\ b > 0 \tag{12}$$

We recall that system (1) is said to be globally asymptotically stable if any of its solutions tends, as $t \to +\infty$, to some equilibrium state /7/.

If $r > 1$ system (12) has three equilibrium states. We can therefore combine the application of Theorem 3 with fairly well-developed estimates for attractors of system (12) /9-11/, thanks to which, for certain parameter values, one can state that an attractor of system (12) is contained in a set $G_1 \cup G_2 \cup \{0\}$, where G_1 and G_2 are disjoint bounded regions each of whose closures contains exactly one equilibrium state.

Here we shall need the following simple assertion.

Lemma. If $b \geqslant 2$, an attractor of system (12) is contained in the set

$$\{z \geqslant 0,\ y^2 + (z-r)^2 \leqslant Br^2,\ y^2 \leqslant Br^2 - 1,\ x^2 \leqslant Br^2 - 1\} \qquad B = b^2/4(b-1) \tag{13}$$

if $b \leqslant 2$, an attractor of system (12) is contained in the set

$$\{z \geqslant 0,\ y^2 + (z-r)^2 \leqslant r^2,\ y^2 \leqslant r^2 - 1,\ x^2 \leqslant r^2 - 1\} \tag{14}$$

The proof follows the same lines as the proof of the analogous result in /12/. When $b \leqslant 2$ we have

$$[y(t)^2 + (z(t)-r)^2 - r^2]^{\cdot} \leqslant -b\,[y(t)^2 + (z(t)-r)^2 - r^2]$$

Hence

$$\overline{\lim_{t\to+\infty}}\,[y(t)^2 + (z(t)-r)^2] \leqslant r^2 \tag{15}$$

The relation $\underline{\lim}_{t\to+\infty} z(t) \geqslant 0$ was proved in /10/.

The fact that the sets $\{y^2 + (z-r)^2 \leqslant r^2, |x| = c\}$ are contact-free for $c > r$ and the estimate (12) imply that $\overline{\lim}_{t\to+\infty} |x(t)| \leqslant r$.

Let $\overline{\lim_{t\to+\infty}}\, |x(t)| \leqslant x_k$. Then it follows from this inequality, the second equation of system (12) and (15), that

$$\overline{\lim_{t\to+\infty}}\, y(t)^2 \leqslant x_k^2 r^2 (1 + x_k^2)^{-1} \tag{16}$$

The fact that the sets $\{y^2 + (z-r)^2 \leqslant r^2,\ y^2 \leqslant x_k^2 r^2 (1 + x_k^2)^{-1}, |x| = c\}$ are contact-free for $c^2 > x_k^2 r^2 (1 + x_k^2)^{-1}$ and the estimates (15) and (16) imply the inequality

$$\overline{\lim}_{t\to+\infty} x(t)^2 \leqslant x_k^2 r^2 (1 + x_k^2)^{-1}$$

Putting $x_{k+1}^2 = x_k^2 r^2/(1 + x_k^2),\ x_0 = r$ and letting $k \to \infty$ in this equality, we obtain $\lim_{k\to\infty} x_k^2 = r^2 - 1$.

This last relation proves the assertion of the lemma when $b \leqslant 2$. When $b \geqslant 2$ the proof proceeds along similar lines.

We now present one of the simplest sufficient conditions for an attractor of system (12) to lie in the set $G_1 \cup G_2 \cup \{0\}$ /11/:

$$\mu^2/4 \geqslant \gamma\sigma_0^2 - 1 \tag{17}$$

$$\mu = \frac{d+1}{\sqrt{d(r-1)}},\quad \gamma = \frac{2d}{b},\quad \sigma_0^2 = \frac{Cr^2 - 1}{2d(r-1)},\quad C = \begin{cases} B & \text{if } b \geqslant 2 \\ 1 & \text{if } b \leqslant 2 \end{cases}$$

If $b = 8/3,\ d = 10$, condition (17) holds for $r \geqslant 4$.

Since $\lambda_1 + \lambda_2 + \lambda_3 = -(d + b + 1)$, a sufficient condition for the assumptions of Theorem 3 to be valid is that $\lambda_3 > -(d + b + 1)$ on the sets (13) and (14). Applying Sylvester's criterion to the matrix

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f^*}{\partial x}\right) - \lambda_3 I$$

we see that the last inequality will hold provided that

$$[(b+1)(b+d)-(d+r-z)^2/4](d+1)-(d+b)y^2/4>0$$

on the sets (13) and (14). This inequality will hold if $b\geqslant 1$,

$$(b+1)(b+d)>\frac{Cr^2(d+b)}{4(d+1)}+\frac{d^2}{4}+\frac{dr}{2} \tag{18}$$

Thus, if conditions (17) and (18) are satisfied, then by Theorem 3 G_1 and G_2 are domains of attraction of stable equilibrium states and, consequently, system (12) will be globally asymptotically stable. Note that for $b=8/3$ and $d=10$, inequality (18) will hold for any $r\leqslant 3.5$. Thus, estimates (17), (18) are somewhat superior to Smith's estimate /13/ in some cases. We also point out that further improvement of the global asymptotic stability conditions obtained here can be achieved by applying the apparatus developed in /14, 15/ for estimating attractors of system (12).

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